

Some Rules of Elementary Transformation in $B^+(E, F)$ and their applications

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Abstract Let E, F be two Banach spaces, $B(E, F)$ the set of all bounded linear operators from E into F , and $B^+(E, F)$ the set of double splitting operators in $B(E, F)$. In this paper, we present some rules of elementary transformations in $B^+(E, F)$, consisting of five theorems. Let $\Phi_{m,n}$ be the set of all Fredholm operators T in $B(E, F)$ with $\dim N(T) = m$ and $\operatorname{codim} R(T) = n$, and $F_k = \{T \in B(E, F) : \operatorname{rank} T = k < \infty\}$. Applying the rules we prove that $F_k (k < \dim F)$ and $\Phi_{m,n} (n > 0)$ are path connected, so that they are not only smooth submanifolds in $B(E, F)$ with tangent space $M(X) = \{T \in B(E, F) : TN(X) \subset R(X)\}$ at X in them, but also path connected. Hereby we obtain the following topological construction: $B(\mathbf{R}^m, \mathbf{R}^n) = \bigcup_{k=0}^n F_k (m \geq n)$, $B(\mathbf{R}^m, \mathbf{R}^n) = \bigcup_{k=0}^m F_k (m < n)$, F_k is path connected and smooth sub-hypersurface in $B(\mathbf{R}^m, \mathbf{R}^n) (k < n)$, and especially $\dim F_k = (m+n-k)k$ for $k = 0, 1, \dots, n$. Finally we introduce an equivalent relation in $B^+(E, F)$ and prove that the equivalent class \tilde{T} generated by T is path connected for any $T \in B^+(E, F)$ with $R(T) \subsetneq F$.

Key words Elementary Transformation Path Connected Set of Operators Dimension Sub-hypersurface Smooth Submanifold Equivalent Relation.

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1 Introduction

Let E, F be two Banach spaces, $B(E, F)$ the set of all linear bounded operators from E into F , and $B^+(E, F)$ the set of all double splitting operators in $B(E, F)$. It is well known that the elementary transformation of a matrix is a power tool in matrix theory. For example, by the elementary transformation one can show that $F_k = \{T \in B(\mathbf{R}^n) : \operatorname{rank} T = k\} (k < n)$ is path connected. When $\dim E = \dim F = \infty$ there is no such elementary transformation in $B(E, F)$, so it is difficult to prove that $F_k (k < \dim F)$ in $B(E, F)$ is path connected. In this paper, we present some rules of elementary transformation in $B^+(E, F)$, which consist of five theorems, see Section 2. Using the rules we prove that $F_k (k < \dim F)$ and $\Phi_{m,n} (n > 0)$ are path connected, where $\Phi_{m,n}$ denotes the set of all Fredholm operators T in $B(E, F)$ with $\dim N(T) = m$ and $\operatorname{codim} R(T) = n$. Then by Theorem 4.2 in [Ma4] we obtain the following result $F_k (k < \dim F)$ and $\Phi_{m,n} (n > 0)$ are not only smooth submanifolds in $B(E, F)$ with tangent space $M(X) =$

$\{T \in B(E, F) : TN(X) \subset R(V)\}$ at any X in then, but also path connected. As its application we have that $B(\mathbf{R}^m, \mathbf{R}^n) = \bigcup_{k=0}^n F_k(m \geq n), B(\mathbf{R}^m, \mathbf{R}^n) = \bigcup_{k=0}^m F_k(m < n), F_k(k < n)$, is path connected and smooth sub-hypersurface in $B(\mathbf{R}^m, \mathbf{R}^n)$ and especially, $\dim F_k = (m + n - k)k$ for $k = 0, 1, \dots, n$. Finally we introduce an equivalent relation in $B^+(E, F)$ and prove that the equivalent class \tilde{T} generated by T is path connected for $T \in B^+(E, F)$ with $R(T) \subsetneq F$.

2 Some Rules of Elementary Transformation in $B^+(E, F)$

In this section, we will introduce some rules of elementary transformation in $B^+(E, F)$, which consist of five theorems. It is useful to imagine the trace of these rules of elementary Transformations from Euclidean space to Banach space.

Theorem 2.1 *If $E = E_1 \oplus R = E_* \oplus R$, then the following conclusions hold:*

(i) *there exists a unique $\alpha \in B(E_*, R)$ such that*

$$E_1 = \{x + \alpha x : \forall x \in E_*\}; \quad (2.1)$$

conversely, for any $\alpha \in B(E_*, R)$ the subspace E_1 defined by (2.1) satisfies $E = E_1 \oplus R$

(ii)

$$P_{E_1}^R = P_{E_*}^R + \alpha P_{E_*}^R \text{ andso } P_R^{E_1} = P_R^{E_*} - \alpha P_{E_*}^R.$$

Proof For the proof of (i) see [Ma3] and [Abr].

Obviously,

$$(P_{E_*}^R + \alpha P_{E_*}^R)^2 = P_{E_*}^R + \alpha P_{E_*}^R, \text{ and}$$

$$P_{E_*}^R x + \alpha P_{E_*}^R x = 0 \text{ for } x \in E \Leftrightarrow P_{E_*}^R x = 0 \Leftrightarrow x \in R$$

Then by (2.1), one concludes

$$P_{E_1}^R = P_{E_*}^R + \alpha P_{E_*}^R, \text{ andso, } P_R^{E_1} = P_R^{E_*} - \alpha P_{E_*}^R.$$

The proof ends. \square

Let $B^+(E)$ be the set of all double splitting operators in $B(E)$ and $C_r(R) = \{T \in B^+(E) : E = R(T) \oplus R\}$

Theorem 2.2 *Suppose that $E = E_* \oplus R$ and $\dim R > 0$. Then $P_{E_*}^R$ and $(-P_{E_*}^R)$ are path connected in the set $\{T \in C_r(R) : N(T) = R\}$.*

Proof Due to $\dim R > 0$, one can assume that $B(E_*, R)$ contains a non-zero operator α , otherwise $E_* = \{0\}$ the theorem is trivial. Let $E_1 = \{x + \alpha x : \forall x \in E_*\}$. Then by Theorem 2.1, $E = E_1 \oplus R$ and

$$P_{E_1}^R = P_{E_*}^R + \alpha P_{E_*}^R. \quad (2.2)$$

Consider the path

$$P(\lambda) = (1 - 2\lambda)P_{E_*}^R + (1 - \lambda)P_{E_*}^R \quad 0 \leq \lambda \leq 1.$$

Clearly,

$$R(P(\lambda)) = R(P_{E_*}^R + \frac{1 - \lambda}{1 - 2\lambda} \alpha P_{E_*}^R), \quad 0 \leq \lambda \leq 1.$$

Then by Theorem 2.1, $R(P(\lambda)) \oplus R = E$, i.e. $P(\lambda) \in C_r(R)$ for all $\lambda \in [0, 1]$. In addition, $P(0) = P_{E_1}^R$, $P(1) = -P_{E_*}^R$ and $N(P(\lambda)) = R, \forall \lambda \in [0, 1]$. This shows that $P_{E_1}^R$ and $-P_{E_*}^R$ are path connected in $\{T \in C_r(R) : N(T) = R\}$.

Next go to show that $P_{E_1}^R$ and $P_{E_*}^R$ are path connected in $\{T \in C_r(R) : N(T) = R\}$. Consider the path

$$P(\lambda) = P_{E_*}^R + \lambda \alpha P_{E_*}^R \quad 0 \leq \lambda \leq 1.$$

By Theorem 2.1, $P(1) = P_{E_*}^R + \alpha P_{E_*}^R = P_{E_1}^R$, and $R(P(\lambda)) \oplus R = E \quad \forall \lambda \in [0, 1]$, where $E_1 = \{x + \alpha x : \forall x \in E_*\}$. Obviously, $N(P(\lambda)) = R$ and $P(0) = P_{E_*}^R$. Therefore $P_{E_1}^R$ and $P_{E_*}^R$ are path connected in $\{T \in C_r(R) : N(T) = R\}$. Thus the theorem is proved. \square

For simplicity, still write $C_r(N) = \{T \in B^+(E, F) : R(T) \oplus N = F\}$ in the sequel.

Theorem 2.3 Suppose $T_0 \in C_r(N)$ and $F = F_* \oplus N$. Then T_0 and $P_{F_*}^N T_0$ are path connected in the set $\{T \in C_r(N) : N(T) = N(T_0)\}$.

Proof One can assume $R(T_0) \neq F_*$, otherwise the theorem is trivial. Then by Theorem 2.1, there exists a non-zero operator $\alpha \in B(F_*, N)$ such that

$$R(T_0) = \{y + \alpha y : \forall y \in F_*\}, \quad P_{R(T_0)}^N = P_{F_*}^N + \alpha P_{F_*}^N,$$

and $P_N^{R(T_0)} = P_N^{F_*} - \alpha P_{F_*}^N$. So

$$T_0 = \{P_{F_*}^N + \alpha P_{F_*}^N\} T_0. \quad (2.3)$$

Let

$$F_\lambda = \{y + \lambda \alpha y : \forall y \in F_*\} \quad \text{for all } \lambda \in [0, 1].$$

Note $\lambda \alpha \in B(F_*, N)$. According to Theorem 2.1 we also have

$$P_{F_\lambda}^N = P_{F_*}^N + \lambda \alpha P_{F_*}^N \quad \text{and} \quad P_N^{F_\lambda} = P_N^{F_*} - \lambda \alpha P_{F_*}^N, \quad \forall \lambda \in [0, 1].$$

Consider the path

$$P(\lambda) = P_{F_\lambda}^N T_0, \quad \forall \lambda \in [0, 1].$$

Because $F = R(T_0) \oplus N = F_\lambda + N$ one observers

$$R(P(\lambda)) = F_\lambda, \quad \forall \lambda \in [0, 1],$$

and so

$$R(P(\lambda)) \oplus N = F, \quad \text{i. e.,} \quad P(\lambda) \in C_r(N).$$

Note

$$y \in N(P(\lambda)) \Leftrightarrow P_{F_\lambda}^N T_0 y = 0 \Leftrightarrow T_0 y \in N \Leftrightarrow y \in N(T_0)$$

i.e., $N(P(\lambda)) = N(T_0), \forall \lambda \in [0, 1]$. Thus

$$P(\lambda) \in \{T \in C_r(N) : N(T) = N(T_0)\}, \quad \lambda \in [0, 1].$$

In addition, $P(1) = T_0$ by (2.3), and $P(0) = P_{F_*}^N T_0$.

Finally we conclude that T_0 and $P_{F_*}^N T_0$ are path connected in the set $\{T \in C_r(N) : N(T) = N(T_0)\}$. The proof ends. \square

Let $C_d(R) = \{T \in B^+(E, F) : E = N(T) \oplus R\}$.

Theorem 2.4 *Suppose that $T_0 \in C_d(R_0)$ and $E = E_* \oplus R_0$. Then T_0 and $T_0 P_{R_0}^{E_*}$ are path connected in the set $\{T \in C_d(R_0) : R(T) = R(T_0)\}$.*

Proof One can assume $E_* \neq N(T_0)$, otherwise the theorem is trivial. Then by Theorem 2.1, there exists a non-zero operator $\alpha \in B(E_*, R_0)$ such that

$$P_{N(T_0)}^{R_0} = P_{E_*}^{R_0} + \alpha P_{E_*}^{R_0},$$

and so,

$$T_0 = T_0 P_{R_0}^{N(T_0)} = T_0 (P_{R_0}^{E_*} - \alpha P_{E_*}^{R_0}). \quad (2.4)$$

Consider the path as follows,

$$P(\lambda) = T_0 (P_{R_0}^{E_*} - \lambda \alpha P_{E_*}^{R_0}), \quad 0 \leq \lambda \leq 1.$$

Since $(P_{R_0}^{E_*} - \lambda \alpha P_{E_*}^{R_0})x = x \quad \forall x \in R_0$, we conclude $R(P(\lambda)) = R(T_0)$. We also have $N(P(\lambda)) = R(P_{R_0}^{E_*} + \lambda \alpha P_{R_0}^{E_*})$. Indeed,

$$x \in N(P(\lambda)) \iff (P_{R_0}^{E_*} - \lambda \alpha P_{E_*}^{R_0})x = 0 \iff x \in R(P_{E_*}^{R_0} + \lambda \alpha P_{E_*}^{R_0})$$

since $\lambda\alpha \in B(E_*, R_0)$ for all $\lambda \in [0, 1]$.

Thus $P(\lambda) \in \{T \in C_d(R) : R(T) = R_0\}, \forall \lambda \in [0, 1]$. In addition, $P(1) = T_0$ by (2.4), and $P(0) = T_0 P_{R_0}^{E_*}$. Then the theorem is proved. \square

Theorem 2.5 *Suppose that the subspaces E_1 and E_2 in E satisfy $\dim E_1 = \dim E_2 < \infty$. Then E_1 and E_2 possess a common complement R , i.e., $E = E_1 \oplus R = E_2 \oplus R$.*

Proof According to the assumption $\dim E_1 = \dim E_2 < \infty$, we have the following decompositions:

$$E = H \oplus (E_1 + E_2), \quad E_1 = E_1^* \oplus (E_1 \cap E_2), \quad \text{and} \quad E_2 = E_2^* \oplus (E_1 \cap E_2).$$

It is easy to observe that $(E_1^* \oplus E_2^*) \cap (E_1 \cap E_2) = \{0\}$ and $\dim E_1^* = \dim E_2^* < \infty$. Indeed, if $e_1^* + e_2^*$ belongs to $E_1 \cap E_2$, for $e_i^* \in E_i^*, i = 1, 2$, then $e_2^* = (e_1^* + e_2^*) - e_1^* \in E_1$ and $e_1^* = (e_1^* + e_2^*) - e_2^* \in E_2$, so that $e_1^* = e_2^* = 0$ because of $E_i^* \cap (E_1 \cap E_2) = \{0\}, i = 1, 2$. Hereby, one can see

$$E_1 + E_2 = (E_1^* \oplus E_2^*) \oplus (E_1 \cap E_2). \quad (2.5)$$

we now are in the position to determine R . We may assume $\dim E_1^* = \dim E_2^* > 0$, otherwise the theorem is trivial. Then $B^X(E_1^*, E_2^*)$ contains an operator α , which bears the subspace H_1 as follows,

$$H_1 = \{x + \alpha x : \forall x \in E_1^*\} = \{x + \alpha^{-1}x : \forall x \in E_2^*\}.$$

By Theorem 2.1,

$$E_1^* \oplus E_2^* = H_1 \oplus E_2^* = H_1 \oplus E_1^*$$

Finally, according to (2.5) we have

$$E = H \oplus (E_1 + E_2) = H \oplus (E_1^* \oplus E_2^*) \oplus (E_1 \cap E_2) = H \oplus H_1 \oplus E_2^* \oplus (E_1 \cap E_2) = H \oplus H_1 \oplus E_2$$

and

$$E = H \oplus H_1 \oplus E_1^* \oplus (E_1 \cap E_2) = H \oplus H_1 \oplus E$$

This says $R = H \oplus H_1$. The proof ends. \square

3 Some Applications

In this section we will give some application of the rules in Section 2.

Theorem 3.1 *For $k < \dim F$, F_k is path connected.*

Proof In what follows, we may assume $k > 0$, otherwise, the theorem is trivial. Let T_1 and T_2 be arbitrary two operators in F_k , and

$$E = N(T_i) \oplus R_i, \quad i = 1, 2.$$

Then $0 < \dim R_1 = \dim R_2 = k < \infty$, and by Theorem 2.5, there exists a subspace N_0 in E such that

$$E = N_0 \oplus R_1 = N_0 \oplus R_2 \quad (3.1)$$

Let

$$L_i x = \begin{cases} T_i x, & x \in R_i \\ 0, & x \in N_0 \end{cases}$$

$i = 1, 2$.

We claim that T_i and L_i are path connected in F_k , $i = 1, 2$. Due to (3.1) the theorem 2.1 shows that there exists an operator $\alpha_i \in B(N_0, R_i)$ such that

$$P_{R_i}^{N(T_i)} = P_{R_i}^{N_0} - \alpha_i P_{N_0}^{R_i}, \quad i = 1, 2.$$

Consider the following paths:

$$P_i(\lambda) = T_i(P_{R_i}^{N_0} - \lambda \alpha_i P_{N_0}^{R_i}), \quad 0 \leq \lambda \leq 1 \text{ and } i = 1, 2.$$

Obviously, $P_i(0) = T_i P_{R_i}^{N_0} = L_i$, and $P_i(1) = T_i(P_{R_i}^{N_0} - \alpha_i P_{N_0}^{R_i}) = T_i P_{R_i}^{N(T_i)} = T_i$, $i = 1, 2$. Next go to show $R(P_i(\lambda)) = T_i R_i = R(T_i)$.

Evidently, $R(P_{R_i}^{N_0} - \lambda \alpha_i P_{N_0}^{R_i}) \subset R_i$, and the converse relation follows from $(P_{R_i}^{N_0} - \lambda \alpha_i P_{N_0}^{R_i})r = r$, $\forall r \in R_i$. This says $P_i(\lambda) \in F_k$ for $0 \leq \lambda \leq 1$ and $i = 1, 2$, so that L_i and T_i are path connected in F_k , $i = 1, 2$. Hence, in what follows, we can assume $N(T_1) = N(T_2) = N_0$. In the other hand, according to Theorem 2.5 we have

$$F = R(T_1) \oplus N_* = R(T_2) \oplus N_* \quad (3.2)$$

where N_* is a closed subspace in F . Then by Theorem 2.3, the proof of the theorem turns to that of the following conclusion: $P_{R(T_1)}^{N_*} T_2$ and T_1 are path connected in F_k . Let

$$T_1^+ y = \begin{cases} (T_1|_{R_1})^{-1} y, & y \in R(T_1) \\ 0, & y \in N_* \end{cases}$$

It is easy to observe

$$P_{R(T_1)}^{N_*} T_2 = \left(P_{R(T_1)}^{N_*} T_2 T_1^+ \right) T_1. \quad (3.3)$$

In fact, note $N(T_1) = N(T_2) = N_0$ and $T_1^+ T_1 = P_{R_1}^{N_0}$, then

$$P_{R(T_1)}^{N_*} T_2 = P_{R(T_1)}^{N_*} T_2 \left(P_{R_1}^{N_0} + P_{N_0}^{R_1} \right) = P_{R(T_1)}^{N_*} T_2 P_{R(T_1)}^{N_0} = P_{R(T_1)}^{N_*} T_2 T_1^+ T_1.$$

We claim $P_{R(T_1)}^{N_*} T_2 T_1^+|_{R(T_1)} \in B^X(R(T_1))$.

Evidently,

$$\begin{aligned} P_{R(T_1)}^{N_*} T_2 T_1^+ y = 0 \text{ for } y \in R(T_1) &\Leftrightarrow T_2 T_1^+ y \in N_* \\ &\Leftrightarrow T_1^+ y \in N_0 \cap R_1 \text{ (because of (3.2))} \\ &\Leftrightarrow T_1^+ y = 0 \Leftrightarrow y = 0, \end{aligned}$$

i.e., $P_{R(T_1)}^{N_*} T_2 T_1^+|_{R(T_1)}$ is injective.

In the other hand, since the decomposition (3.2) implies $P_{R(T_1)}^{N_*}|_{R(T_2)} \in B^X(R(T_2), R(T_1))$, there is a unique $r_2 \in R_2$ for any $y \in R(T_1)$, such that

$$y = P_{R(T_1)}^{N_*} T_2 r_2 = P_{R(T_1)}^{N_*} T_2 \left(P_{R_1}^{N_0} r_2 + P_{N_0}^{R_2} r_2 \right) = P_{R(T_1)}^{N_*} T_2 P_{R_1}^{N_0} r_2$$

Then $y_0 = T_1 P_{R_1}^{N_0} r_2$ fulfills

$$y = P_{R(T_1)}^{N_*} T_2 P_{R_1}^{N_*} r_2 = P_{R(T_1)}^{N_*} T_2 T_1^+ y_0.$$

This says that $P_{R(T_1)}^{N_*} T_2 T_1^+|_{R(T_1)}$ is surjective. So $P_{R(T_1)}^{N_*} T_2 T_1^+|_{R(T_1)} \in B^X(R(T_1))$ of all invertible operators in $B(R(T_1))$.

It is well known that $P_{R(T_1)}^{N_*} T_2 T_1^+|_{R(T_1)}$ is path connected with some one of $-I_{R(T_1)}$ and $I_{R(T_1)}$ is $B^X(R(T_1))$, where $I_{R(T_1)}$ denotes the identity on $R(T_1)$. Let $Q(t)$ be a path in $B^X(R(T_1))$ with $Q(0) = P_{R(T_1)}^{N_*} T_2 T_1^+|_{R(T_1)}$ and $Q(1) = -I_{R(T_1)}$ (or $I_{R(T_1)}$). Then $Q_1(t) = Q(t)T_1$ is a path in the set $S = \{T \in B(E, F) : R(T) = R(T_1) \text{ and } N(T) = N(T_1)\}$ satisfying

$$Q_1(0) = P_{R(T_1)}^{N_*} T_2 T_1^+ T_1 \quad \text{and} \quad Q_1(1) = -T_1 \text{ (or } T_1).$$

Since $S \subset F_k$, it follows that $P_{R(T_1)}^{N_*} T_2 T_1^+ T_1$ and $-T_1$ (or T_1) are path connected in F_k . By (3.3) we merely need to show the following conclusion: if $P_{R(T_1)}^{N_*} T_2 T_1^+ T_1$ is path connected with $-T_1$ in F_2 then $P_{R(T_1)}^{N_*} T_2 T_1^+ T_1$ and T_1 are path connected in F_k . Note $F = R(T_1) \oplus N_*$ and $\dim N_* > 0$. By Theorem 2.2, $P_{R(T_1)}^{N_*}$ and $-P_{R(T_1)}^{N_*}$ are path connected in the set $\{T \in$

$G_r(N_*) : N(T) = N_*\} = \{T \in B(F) : F = R(T) \oplus N_* \text{ and } N(T) = N_*\}$. Let $Q(t)$ be such a path in the set with $Q(0) = -P_{R(T_1)}^{N_*}$ and $Q(1) = P_{R(T_1)}^{N_*}$. Since $F = R(Q(t)) \oplus N_*$ and $N(Q(t)) = N_* \forall t \in [0, 1]$, $R(Q(t)T_1) = Q(t)R(T_1) = Q(t)F$, $Q(t)$ for $t \in [0, 1]$ belongs to F_k , and satisfies $Q(0)T_1 = -P_{R(T_1)}^{N_*}T_1 = -T_1$ and $Q(1)T_1 = T_1$. This says that $-T_1$ and T_1 are path connected in F_k , so that the conclusion is proved. \square

It is obvious that if either $\dim E$ or $\dim F$ is finite, then $B(E, F)$ consists of all finite rank operators. Hence we assume $\dim E = \dim F = \infty$ in the sequel.

Let $\Phi_{m,n} = \{T \in B^+(E, F) : \dim N(T) = m < \infty \text{ and } \text{codim } R(T) = n < \infty\}$, we have

Theorem 3.2 $\Phi_{m,n}, (n > 0)$ is path connected.

Proof Let T_1 and T_2 be arbitrary two operators in $\Phi_{m,n}$ and

$$F = R(T_1) \oplus N_1 = R(T_2) \oplus N_2,$$

i.e. $T_1 \in C_r(N_1)$ and $T_2 \in C_r(N_2)$.

Clearly, $\dim N_1 = \dim N_2 = n < \infty$. Then by Theorem 2.5, there exists a subspace F_* in F such that

$$F = R(T_1) \oplus N_1 = F_* \oplus N_1 \text{ and } F = R(T_2) \oplus N_2 = F_* \oplus N_2. \quad (3.4)$$

So, by Theorem 2.3 the proof of the theorem turns to that of the operators $P_{F_*}^{N_1}T_1$ and $P_{F_*}^{N_2}T_2$ being path connected in $\Phi_{m,n}$. For simplicity, still write $P_{F_*}^{N_1}T_1$ and $P_{F_*}^{N_2}T_2$ as T_1 and T_2 , respectively. However, here $R(T_1) = R(T_2) = F_*$ while $N(T_1), N(T_2)$ keep invariant. Due to $\dim N(T_1) = \dim N(T_2) = m < \infty$. Theorem 2.5 shows that there exists a subspace R in E such that

$$E = N(T_1) \oplus R = N(T_2) \oplus R. \quad (3.5)$$

Then by Theorem 2.4, one can conclude that $T_2 P_R^{N(T_1)}$ and T_2 are path connected in $\Phi_{m,n}$. Thus the proof of the theorem turns once more to that of $T_2 P_R^{N(T_1)}$ and T_1 being path connected in $\Phi_{m,n}$. In what follows, we do this.

Let

$$T_1^+ y = \begin{cases} (T_1|_R)^{-1} y, & y \in F_*, \\ 0, & y \in N_1. \end{cases}$$

Then

$$T_2 P_R^{N(T_1)} = T_2 T_1^+ T_1. \quad (3.6)$$

It is easy to observe $T_2T_1^+|_{F_*} \in B^X(F_*)$. Indeed,

$$\begin{aligned} T_2T_1^+y = 0 \text{ for } y \in F_* &\Leftrightarrow T_1^+y \in N(T_2) \cap R \\ &\Leftrightarrow T_1^+y = 0 \text{ because of (3.5)} \Leftrightarrow y = 0, \end{aligned}$$

i.e., $T_2T_1^+|_{F_*}$ is injective; let; $r \in R$ satisfy $T_2r = y$ for any $y \in F_*$, and $T_1r - y_0 \in F_*$, then

$$T_2T_1^+y_0 = T_2T_1^+T_1r = T_2P_R^{N(T_1)}r = T_2r = y,$$

and so $T_2T_1^+|_{F_*}$ is also surjective. Thus $T_2T_1^+|_{F_*} \in B^X(F_*)$ and is path connected with some one of $-I_{F_*}$ and I_{F_*} is in $B^X(F_*)$. Similar to the way in the proof of Theorem 3.1 one can prove that $T_2T_1^+T_1$ is path connected with some one of $-T_1$ and T_2 is in $\Phi_{m,n}$. Note the equality $F = F_* \oplus N_1$ in (3.4) and $\dim N_1 > 0$. Due to $\dim N_1 > 0$ Theorem 2.2 shows that $-P_{F_*}^{N_1}$ and $P_{F_*}^{N_1}$ are path connected in the set $S = \{T \in C_F(N_1) \subset (F); N(T) = N_1\}$, say that $Q(X) \in S$ for $\lambda \in [0, 1]$ fulfills $Q(0) = -P_{F_*}^{N_1}$ and $Q(1) = P_{F_*}^{N_1}$. Obviously,

$$R(Q(\lambda)T_1) = Q(\lambda)F_* = Q(\lambda)(F_* \oplus N_1) = R(Q(\lambda)) \quad (\text{note } R(T_1) = F_*),$$

and

$$Q(\lambda)T_1x = 0 \Leftrightarrow T_1x \in N_1 \Leftrightarrow x \in N(T_1) \quad (\text{note } F_* \cap N_1 = \{0\})$$

for all $\lambda \in [0, 1]$. So $Q(\lambda)T_1$ for any $\lambda \in [0, 1]$ belongs to $\Phi_{m,n}$. This says that $-T_1$ and T_1 are path connected in $\Phi_{m,n}$. Therefore by (3.6) $T_2P_R^{N(T_1)}$ is also path connected with T_1 in $\Phi_{m,n}$. \square

By Theorem 4.2 in [Ma4] we further have

Theorem 3.3 $F_k(k < \dim F)$ and $\Phi_{m,n}(n > 0)$ are not only path connected set but also smooth submanifolds in $B(E, F)$ with tangent space $M(X) = \{T \in B(E, T) : TN(X) \subset R(X)\}$ at any X in them.

Applying the theorem to $B(\mathbf{R}^m, \mathbf{R}^n)$ we obtain the geometrical and topological construction of $B(\mathbf{R}^m, \mathbf{R}^n)$.

Theorem 3.4 $B(\mathbf{R}^m, \mathbf{R}^n) = \bigcup_{k=0}^n F_k(m \geq n), B(\mathbf{R}^m, \mathbf{R}^n) = \bigcup_{k=0}^n F_k(m < n), F_k$ is a smooth and path connected submanifold in $B(\mathbf{R}^m, \mathbf{R}^n)$, and especially, $\dim F_k = (m + n - k)k$ for $k = 0, 1, \dots, n$.

Proof We need only to prove the formula $\dim F_k = (m + n - k)k$ for $k = 0, 1, \dots, n$ since otherwise the theorem is immediate from Theorem 3.3. Let $T = \{T_{i,j}\}_{i,j=1}^{m,n}$ for any $T \in B(\mathbf{R}^m, \mathbf{R}^n), I_k = \{T \in B(\mathbf{R}^m, \mathbf{R}^n) : t_{i,j} = 0 \text{ except } t_{i,i} = 1, 1 \leq i \leq k\}$, and $I_k^+ = \{(s_{i,j})_{i,j=1}^{m,n} \in B(\mathbf{R}^n, \mathbf{R}^m) : s_{i,j} = 0 \text{ except } s_{i,i} = 1, 1 \leq i \leq k\}$.

Obviously, $I_k I_k^+ I_k = I_k$ and $I_k^+ T_k T_k^+ = T_k^+$. So

$$P_{N(I_k^+)}^{R(I_k)} = I_n^* - I_k I_k^+ = \{\{s_{i,j}\}_1^n \in B(\mathbf{R}^n) : s_{i,j} = 0 \text{ except } s_{i,i} = 1, k+1 \leq i \leq n\},$$

and

$$P_{N(I_k)}^{R(I_k^+)} = I_m^* - I_k^+ I_k = \{\{s_{i,j}\}_1^m \in B(\mathbf{R}^m) : s_{i,j} = 0 \text{ except } s_{i,i} = 1, k+1 \leq i \leq m\},$$

where I_n^*, I_m^* denote the identity on \mathbf{R}^n and \mathbf{R}^m , respectively. Let

$$M^+ = \{P_{N(I_k^+)}^{R(I_k)} T P_{N(I_k)}^{R(I_k^+)} : \forall T \in B(\mathbf{R}^m, \mathbf{R}^n)\}.$$

By direct computing

$$M^+ = \{\{\lambda_{i,j}\}_{i,j=1}^{n,m} : \lambda_{i,j} = 0 \text{ except } \lambda_{i,i} = t_{i,i}, k+1 \leq i \leq n \text{ and } k+1 \leq j \leq m\}.$$

so that $\dim M^+ = (n-k)(m-k)$. By Lemma 4.1 in [Ma4], $B(\mathbf{R}^m, \mathbf{R}^n) = M(I_k) \oplus M^+$ and so, $\dim M(I_k) = m \times n - (m-k)(n-k) = (m+n-k)k$. Due to F_k being path connected, by Theorem 3.1, one can conclude $\dim F_k = \dim M(I_k) = (m+n-k)k$. \square

Let $G(\cdot)$ denote the set of all splitting subspaces in the Banach space in the parentheses, $U_E(R) = \{H \in G(E) : E = R \oplus H\}$ for any $R \in G(E)$, and $U_F(S) = \{L \in G(E) : F = S \oplus L\}$ for any $S \in G(E)$. In order to consider of more general results then that of the previous theorems 3.1 and 3.2, we introduce the equivalent relation as follows.

Definition 3.1 T_0 and T_* in $B^+(E, F)$ are said to be equivalent provided there exist finite number of subspaces N_1, \dots, N_m in $G(E)$, and F_1, \dots, F_n in $G(F)$ such that all

$$U_E(N(T_0)) \cap U_E(N_1), \dots, U_E(N_m) \cap U_E(N(T_*))$$

and

$$U_F(R(T_0)) \cap U_F(F_1), \dots, U_F(F_n) \cap U_F(R(T_*))$$

are non-empty. For abbreviation, write it as $T_0 \sim T_*$, and let \widetilde{T} denote the equivalent class generated by T in $B(E, F)$.

Theorem 3.5 $\widetilde{T_0}$ for any $T_0 \in B^+(E, F)$ with $\text{codim} R(T_0) > 0$ is path connected.

Proof Assume that T_* is any operator in $\widetilde{T_0}$, and

$$R_1 \in U_E(N(T_0)) \cap U_E(N_1), \dots, R_m \in U_E(N_{m-1}) \cap U_E(N_m), R_{m+1} \in U_E(N_m) \cap U_E(N(T_*)). \quad (3.7)$$

We define by induction:

$$T_k = T_{k-1}P_{R_k}^{N_k}, \quad k = 1, 2, \dots, m.$$

It is easy to observe

$$R(T_k) = R(T_0) \text{ and } N(T_k) = N_k, k = 1, \dots, m. \quad (3.8)$$

Indeed, $R(T_1) = T_0R(P_{R_1}^{N_1}) = T_0R_1 = R(T_0)$ and $N(T_1) = \{x \in E : P_{R_1}^{N_1}x \in N(T_0)\} = \{x \in E : P_{R_1}^{N_1}x = 0\} = N_1$ because of $E = R_1 \oplus N_1$; similarly, by induction one can conclude $R(T_k) = R(T_0)$ and $N(T_k) = N_k$ for $k = 1, 2, \dots, m$. So $T_k \in \tilde{T}_0$ for $k = 1, 2, \dots, m$.

Let $S_k = \{T \in G_d(R_k) : R(T) = R(T_0)\} = \{T \in B(E, F) : E = N(T) \oplus R_k \text{ and } R(T) = R(T_0)\}$ for $k = 1, 2, \dots, m$. Evidently $S_k \subset \tilde{T}_0, k = 1, 2, \dots, m$. In fact, by (3.7) $R_1 \in U_E(N(T_0)) \cap U_E(N_1), \dots, R_k \in U_E(N_{k-1}) \cap U_E(N_k)$; while $R_k \in U_E(N_k) \cap U_E(N(T))$ and $R(T) = R(T_0)$; so that $S_k \subset \tilde{T}_0, k = 1, 2, \dots, m$. Next go to show that T_0 and T_m are path connected in \tilde{T}_0 . Due to $N_{k-1} \oplus R_k = N(T_{k-1}) \oplus R_k = N_k \oplus R_k$. Theorem 2.4 shows that T_{k-1} and T_k are path connected in S_k , and so in \tilde{T}_0 for $k = 1, 2, \dots, m$. Therefore T_0 and T_m are path connected in \tilde{T}_0 .

Let $M_k = \{T \in C_d(R_k) : R(T) = R(T_k)\}$. By (3.8)

$$M_k = R(T_k) = \{T \in C_d(R_k) : R(T) = R(T_0)\}, \quad k = 1, 2, \dots, m,$$

so that $M_k \subset \tilde{T}_0$. Note $E = N(T_{k-1}) \oplus R_k = N_k \oplus R_k$. Then by Theorem 2.4, $T_k = T_{k-1}P_{R_k}^{N_k}$ and T_{k-1} are path connected in $M_{k-1} = \{T \in C_d(R_k) : R(T) = R(T_{k-1})\} \subset \tilde{T}_0, k = 1, 2, \dots, m$. This shows that T_0 and T_m are path connected in \tilde{T}_0 . Finally go to prove that T_m and T_* are path connected in \tilde{T}_0 .

In other hand, assume

$$S_1 \in U_F(R(T_0)) \cap U_F(F_1), \dots, S_n \in U_F(F_{n-1}) \cap U_F(F_n), S_{n+1} \in U_F(F_n) \cap U_F(R(T_*)). \quad (3.9)$$

Write $T_{m,0} = T_m$. We define by induction,

$$T_{m,i} = P_{F_i}^{S_i} T_{m,i-1}, \quad i = 1, 2, \dots, n.$$

According to the equality $R(T_m) = R(T_0)$ in (3.8), we infer

$$R(T_{m,i}) = F_i \text{ and } N(T_{m,i}) = N(T_m), \quad i = 1, \dots, n. \quad (3.10)$$

In fact wsmite $F_0 = \mathbf{R}(T_m)$, then by (3.9)

$$F = F_{i-1} \oplus S_i = F_i \oplus S_i, i = 1, 2, \dots, n;$$

and so

$$N(T_{m,i}) = N(T_m) \quad \text{and} \quad R(T_{m,i}) = F_i, i = 1, 2, \dots, n.$$

Thus, take $T_{m,i-1}, F_i$ and S_i for $i = 1, 2, n$ in the places of T_0, F_* and N , in Theorem 2.3, respectively, then the theorem shows that $T_{m,k}$ and $T_{m,k-1}$ are path connected in $\{T \in C_r(S_k) : N(T) = N_m\} \subset \widetilde{T_0}, k = 1, 2, \dots, n$, so that T_m and $T_{m,n}$ are path connected in $\widetilde{T_0}$. Since T_0 and T_m are path connected in $\widetilde{T_0}$, T_0 and $T_{m,n}$ are path connected in $\widetilde{T_0}$, so the proof of the theorem reduces to that of $T_{m,n}$ being path connected with T_* in $\widetilde{T_0}$, where $T_{m,n}$ and T_* satisfy

$$E = N(T_{m,n}) \oplus R_{m+1} = N(T_*) \oplus R_{m+1} \text{ because of } N(T_{m,n}) = N_m,$$

and

$$F = R(T_{m,n}) \oplus S_{n+1} = R(T_*) \oplus S_{n+1} \text{ because of } R(T_{m,n}) = F_n. \quad (3.11)$$

The first equality in (3.11) shows $T_* \in C_\alpha(R_{m+1})$ and $E = N_m \oplus R_{m+1}$. Then by Theorem 2.4, $T_* P_{R_{m+1}}^{N_m}$ and T_* are path connected in $\{T \in C_\alpha(R_{m+1}) : R(T) = R(T_*)\} \subset \widetilde{T_0}$.

The second equality in (3.11) shows $T_* \in C_r(S_{n+1})$ and $F = F_n \oplus S_{n+1}$. Then by Theorem 2.3, $P_{F_n}^{S_{n+1}} T_*$ and T_* are path connected in $\{T \in C_r(S_{n+1}) : N(T) = N(T_*)\} \subset \widetilde{T_0}$. Combining the preceding results, we conclude that $P_{F_n}^{S_{n+1}} T_* P_{R_{m+1}}^{N_m}$ and T_* are path connected in $\widetilde{T_0}$. So far, the proof of the theorem reduces to that of $P_{F_n}^{S_{n+1}} T_* P_{R_{m+1}}^{N_m}$ being path connected with $T_{m,n}$ in $\widetilde{T_0}$.

According to (3.10) we have

$$E = N_m \oplus R_{m+1} \text{ and } F = F_n \oplus S_{n+1}$$

where $N_m = N(T_{m,n})$ and $R(T_{m,n}) = F_n$.

Let

$$T_{m,n}^+ y = \begin{cases} (T_{m,n}|_{R_{m+1}})^{-1} y, & y \in F_n, \\ 0, & y \in S_{n+1}. \end{cases}$$

Obviously,

$$P_{F_n}^{S_{n+1}} T_* P_{R_{m+1}}^{N_m} = P_{F_n}^{S_{n+1}} T_* T_{m,n}^+ T_{m,n}. \quad (3.12)$$

We claim

$$P_{F_n}^{S_{n+1}} T_* T_{m,n}^+|_{F_n} \in B^X(F_n)$$

$P_{F_n}^{S_{n+1}} T_* T_{m,n}^+ y = 0$ for $y \in F_n \Leftrightarrow T_* T_{m,n}^+ y = 0$ because of (3.11) $\Leftrightarrow T_{m,n}^+ y = 0$ since $T_{m,n}^+ y \in R_{m+1} \Leftrightarrow y = 0$; while from the assumption in (3.7) and (3.8), $R_{m+1} \in U_E(N_m) \cap U_E(N(T_*))$ and $S_{n+1} \in U_F(F_n) \cap U_F(R(T_*))$, it follows that there exist $r \in R_{m+1}$ and $y_0 \in F_n$, for any $y \in F_n$ such that $P_{F_n}^{S_{n+1}} T_* r = y$ and $T_{m,n}^+ y_0 = r$, i.e., $P_{F_n}^{S_{n+1}} T_* T_{m,n}^+$ is surjective.

It is well known that $P_{F_n}^{S_{n+1}} T_* T_{m,n}^+|_{F_n}$ is path connected with some one of I_{F_n} and $(-I_{F_n})$ in $B^X(F_n)$. Hence $P_{F_n}^{S_{n+1}} T_* T_{m,n}^+ P_{F_n}^{S_{n+1}}$ is path connected with some one $P_{F_n}^{S_{n+1}}$ and $(-P_{F_n}^{S_{n+1}})$ in the set $S = \{F \in B(F_n) : R(T) = F_n \text{ and } N(T) = S_{n+1}\} \subset \{T \in C_r(S_{n+1}) : N(T) = S_{n+1}\}$. If $P_{F_n}^{S_{n+1}} T_* T_{m,n}^+ P_{F_n}^{S_{n+1}}$ is path connected with $(-P_{F_n}^{S_{n+1}})$, then by Theorem 2.2, it is path connected with $P_{F_n}^{S_{n+1}}$ in S . Thus $P_{F_n}^{S_{n+1}} T_* T_{m,n}^+ P_{F_n}^{S_{n+1}}$ is also path connected with $P_{F_n}^{S_{n+1}}$ in $\{T \in C_r(S_{n+1}) : N(T) = S_{n+1}\}$. Note the equality (3.10). Finally, by the similar way to that in the proof of Theorem 3.2, one can infer that $P_{F_n}^{S_{n+1}} T_* P_{R_{m+1}}^{N_m}$ and T_m are path connected in \tilde{T}_0 . The proof ends. \square

It is easy to see that $\tilde{T} = F_k$ for any $T \in F_k (k < \infty)$ as well as $\tilde{T} = \Phi_{m,n}$ for any $T \in \Phi_{m,n}, n > 0$.

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